Broadcast with Hitch-hiking in

Wireless Ad-Hoc Networks

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Most results appeared in MASS 2005.
Broadcast in Wireless Networks

Figure 1: Reducing energy consumption through relaying.
Broadcast in Wireless Networks

Input: $G = (V, E, c)$ weighted directed graph on network nodes with a power requirement function $c : E \to R^+$ defined on the arcs, and a root $r \in V$

Output: A power assignment function $p : V \to R^+$. A directed arc $(u, v)$ is supported by $p$ if $p(u) \geq c(u, v)$. The supported subgraph must contain a path from $r$ to every other node.

Objective: Minimize $\sum_{v \in V} p(v)$
Figure 2: Node $r$ can reach $v_1$ and $v_2$ with one single transmission. Node $v_1$ relays to $v_3$.
Existing Algorithms for Broadcast

Best approximation ratio: $2 \log n$ (Calinescu et. al. ESA 2003.)

Interesting special cases:

1. power requirements symmetric ($c(u,v) = c(v,u)$) - same algorithm

2. Euclidean case: $c(u,v) = d(u,v)^\kappa$, where $d(u,v)$ the Euclidean distance between $u$ and $v$ and $\kappa$ is the signal attenuation exponent, which is assumed to be in between 2 and 5 and is the same for all pairs of nodes - Minimum Spanning Tree algorithm has approximation ratio 6 (Ambuehl, ICALP 2005)

3. Line case: Euclidean case when all nodes lie on a single line - has polynomial-time algorithms
Hitch-hiking advantage

Figure 3: An illustration of the advantage of hitch-hiking. Node $v_3$, who is equipped with a maximal ratio combiner, receives $1/3$ of the information from $s$ and $2/3$ from $v_1$. 
Formal Definition

The input consists of a complete directed graph $G = (V, E)$ with power requirement function $c : E \to R^+$, and a source $s \in V$.

The output consists of a permutation $\tau = < v_1, v_2, \ldots, v_n >$ of $V$ with $v_1 = s$ and power assignment $p(v)$ of every vertex $v$. For every $1 \leq i < j \leq n$, define $q(v_i v_j) = p(v_i) / c(v_i v_j)$. An output is feasible if for every $j > 1$ we have $\sum_{i=1}^{j-1} q(v_i v_j) \geq 1$. The objective is to minimize $\sum_{i=1}^{n} p(v_i)$.

Note: This is an idealized version!
Finding the permutation is hard!

Note that once the permutation $\tau$ is given, the best values for $p(v_i)$ are given by solving a linear program with variable $p(v_i)$ for all $i$ and $q(v_i v_j)$ for all $i < j$. We call this linear program $LP_1$:

Minimize $\sum_{v \in V} p(v)$ subject to

\begin{align*}
q(v_i v_j) c(v_i v_j) &\leq p(v_i) \quad \forall 1 \leq i < j \leq n \tag{1} \\
\sum_{i=1}^{j-1} q(v_i v_j) &\geq 1 \quad \forall j \geq 2. \tag{2} \\
q(v_i v_j) &\geq 0 \quad \forall 1 \leq i < j \leq n \tag{3} \\
p(v_i) &\geq 0 \quad \forall 1 \leq i \leq n \tag{4}
\end{align*}
Previous Results

- Maric and Yates, JSAC 2004: NP-Hardness and heuristics/simulation
- Manish Agarwal, Joon Ho Cho, Lixin Gao, and Jie Wu: INFOCOM 2004 - heuristics and simulation: up to 50% savings versus No-Hitch-hiking Broadcast

Our Results

- Theorem 1: $OPT_B \leq c \log^2 n \ OPT_{HB}$
- Theorem 2: there are instances with $OPT_B \geq c' \log^2 n \ OPT_{HB}$
- Such examples exist for also for Unicast!
- Line case: $OPT_B \leq 2 \frac{\pi^2}{2 - \frac{\pi^2}{6}} OPT_{HB}$
Proof of Theorem 1

The proof is based on a probabilistic argument. A new power function $\overline{p}(v)$ for $v \in V$ is constructed. For $1 \leq i < j \leq n$, edges $v_iv_j$ are said to be selected when $\overline{p}(v_i) \geq c(v_iv_j)$, or in other words the packet sent by $v_i$ would be decoded completely by $v_j$. For $j > 1$, we call the vertex $v_j$ covered if there some $i < j$ with $v_iv_j$ selected. One can easily check that if every vertex is covered, then we have a feasible solution to the Min-Energy Broadcast problem.

We assign power $\overline{p}_i = \overline{p}(v_i)$ as follows: for each $i$, independently pick $x_i \in (0, 1)$.

1. If $x_i < \frac{1}{2n}$, set $\overline{p}_i = 2p_in$.
2. If $x_i \geq \frac{1}{2n}$, set $\overline{p}_i = \frac{2p_i}{x_i}$.
Proof of Theorem 1 - Continued

Figure 4: A solution with hitch-hiking
Figure 5: "Randomly" we pick $x_r = 0.4$. Then $\bar{p}_r = \frac{2p_r}{0.4} = 50$. 

Proof of Theorem 1 - Continued
Proof of Theorem 1 - Continued

Figure 6: "Randomly" we pick $x_u = 0.8$. Then $\bar{p}_u = \frac{2p_u}{0.8} = 17.5$. 
Proof of Theorem 1 - Continued

Figure 7: "Randomly" we pick $x_v = 0.9$. Then $\bar{p}_v = \frac{2p_v}{0.9} = 40$. 
Proof of Theorem 1 - Continued

Figure 8: ”Randomly” we pick \( x_x = 0.5 \). Then \( \overline{p}_x = \frac{2p_x}{0.5} = 40 \).
Proof of Theorem 1 - Continued

Then the expected value

$$E[\bar{p}_i] = \frac{1}{2n} \cdot 2p_i n + \int_{1/2n}^{1} \frac{2p_i}{x_i} = p_i + 2p_i (\ln 1 - \ln \frac{1}{2n}) = p_i (1 + 2 \ln 2n),$$

and thus the total expected power of the random power assignment \(\bar{p}\) is

$$E[\sum_{i=1}^{n} \bar{p}_i] = \sum_{i=1}^{n} E[\bar{p}_i] = (1 + 2 \ln 2n) \sum_{i=1}^{n} p_i.$$
If nodes $u$ and $v$ satisfy $\frac{1}{2n} \leq q(uv) \leq \frac{1}{2}$, we have that the probability edge $uv$ is selected is equal to

$$Pr[\bar{p}(u) \geq c_{uv}] \geq Pr[\bar{p}(u) \geq \frac{p(u)}{q(uv)}] = Pr[\frac{2p(u)}{x_u}]$$

$$\geq \frac{p(u)}{q(uv)} = Pr[x_u \leq 2q(uv)] = 2q(uv),$$

where we used the constraint $q(uv) \leq \frac{p(u)}{c_{uv}}$, the fact that $\frac{p(u)}{q(uv)} \leq 2np(u)$ and the fact that $x_u$ is uniformly picked from the interval $(0, 1)$. Thus for every $u$ with $u$ before $v$ in $\tau$, the probability $uv$ is selected is at least:

$$z_{uv} = \begin{cases} 
1 & \text{if } q(uv) > \frac{1}{2} \\
2q(uv) & \text{if } \frac{1}{2n} \leq q(uv) \leq \frac{1}{2} \\
0 & \text{if } q(uv) < \frac{1}{2n}
\end{cases}$$
Proof of Theorem 1 - Continued

From now on we follow the exposition from Vazirani’s book closely in computing the probability that a node \( v \) is covered by a selected edge. Since \( \sum_u q(uv) \geq 1 \), and we have \( n \) nodes, 
\[
\sum_{u:q(uv)\geq \frac{1}{2n}} q(uv) \geq \frac{1}{2}
\]
and therefore \( \sum_u z_{uv} \geq 1 \). Using elementary calculus, it is easy to show that under this condition, the probability that \( v \) is covered by selected edges is minimized when each of the \( z_{uv} \) is \( 1/n \). Using the fact that each vertex \( u \) independently chooses \( \overline{p}(u) \), we have:

\[
Pr[v \text{ is covered by selected edges}] \geq 1 - \left(1 - \frac{1}{n}\right)^n \geq 1 - \frac{1}{e},
\]

where \( e \) is the base of natural logarithms. Hence each node is covered with constant probability by the selected edges.
Proof of Theorem 1 - Continued

for every $u$, and among them use the largest value when computing $\bar{p}(u)$, where $c$ is a constant such that

\[
\left( \frac{1}{e} \right)^{c \log n} \leq \frac{1}{4n}.
\]

Now,

\[
Pr[\text{v is not covered by selected edges}] \leq \left( \frac{1}{e} \right)^{c \log n} \leq \frac{1}{4n}.
\]

Summing over all nodes $v \in V$, we get

\[
Pr[\text{there is vertex } v \text{ not covered by selected edges}] \leq n \cdot \frac{1}{4n} \leq \frac{1}{4}.
\]
Proof of Theorem 1 - Continued

We obtain $E[\sum_{i=1}^{n} \bar{p}_i] \leq c \log n (1 + 2 \ln 2n) \sum_{i=1}^{n} p_i$. Applying Markov’s Inequality we obtain

$$Pr[\sum_{i=1}^{n} \bar{p}_i \geq 4c \log n (1 + 2 \ln 2n) \sum_{i=1}^{n} p_i] \leq \frac{1}{4}. $$

The probability of the union of the two undesirable events is $\leq 1/2$. Hence,

$$Pr[\sum_{i=1}^{n} \bar{p}_i \leq 4c \log n (1+2 \ln 2n) \sum_{i=1}^{n} p_i \text{ and every } v \text{ is covered by selected edges}] \geq \frac{1}{2}. $$

Thus there exist a valid power assignment without hitch-hiking of total power at most $4c \log n (1 + 2 \ln 2n) \sum_{i=1}^{n} p_i$, completing the proof of the theorem.
**Idea of Theorem 2**

There are instances with $OPT_B \geq c' \log^2 n \OPT_{HB}$

Construction from Vazirani’s book: For a positive $k$, we get a bipartite graph $H$ with the vertex set $V(H) = A \cup B$, $A, B$ disjoint, and $n/2 = |A| = |B| = 2^k - 1$; every vertex of $B$ has degree $2^{k-1}$.

Moreover, any subset $S \subseteq A$ ”covering” $B$ must have at least $k = \Theta(\log n)$ vertices. We reproduce this construction many times.
Idea of Theorem 2

This only gives instance with $OPT_B \geq c' \log n$  $OPT_{HB}$

Figure 9: In Broadcast without Hitch-Hiking, $\Theta(\log n)$ nodes of $A$ must have power $n$; with Hitch-Hiking every node of $A$ has power 4 as every node of $B$ has degree $n/4$. 
The Line Case

Figure 10: We believe this is gives the largest $OPT_B/OPT_{BH}$ in the Line case - the ratio would be $1 + 1/4 + 1/9 + \ldots + 1/n^2 \leq \pi^2/6$. 
Approximation Algorithms

The known algorithms for Broadcast achieve for Broadcast with Hitch-Hiking:

1. approximation ratio $O(\log^3 n)$ with arbitrary asymmetric power requirements

2. approximation ratio $O(\log^2 n)$ in the Euclidean case;

3. constant approximation ratio in the Line case

Also, for Unicast $O(\log^2 n)$ approximation ratio.
Conclusions and Open Problems

We achieved:

1. Tight, up to a constant, bounds of $OPT_B/OPT_{HB}$ in the case of arbitrary power requirements and in the line case

2. Approximation algorithms for Broadcast with Hitch-Hiking

Left Open:

1. A realistic model of Hitch-Hiking

2. Ratio of $OPT_B/OPT_{HB}$ in the Euclidean case

3. Polynomial-time algorithm for Broadcast with Hitch-Hiking in the Line case