A note on the MST heuristic for bounded edge-length Steiner Trees with minimum number of Steiner Points

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Abstract

We give a tight analysis of the MST heuristic recently introduced by G.-H. Lin and G. Xue for approximating the Steiner tree with minimum number of Steiner points and bounded edge-lengths. The approximation factor of the heuristic is shown to be one less than the *MST number* of the underlying space, defined as the maximum possible degree of a minimum-degree MST spanning points from the space. In particular, on instances drawn from the rectilinear (resp. Euclidean) plane, the MST heuristic is shown to have tight approximation factors of 3, respectively 4.

Keywords: Approximation algorithms, Steiner trees, MST heuristic, VLSI CAD, fixed wireless network design.

1 Introduction

The classical Steiner tree problem is that of finding a shortest tree spanning a given set of terminal points. The tree may use additional points besides the terminals, these points are commonly referred to as *Steiner* points. In the *Minimum number of Steiner Points Tree (MSPT) problem* one also seeks a tree spanning all terminals, but there is an upper-bound on the length of tree edges, and the objective is to minimize the number of Steiner points.

The MSPT problem was first introduced by Sarrafzadeh and Wong [8], motivated by applications to VLSI CAD and network design. In these applications terminals are typically points in the plane, and the underlying metric is either L_1, as in buffer insertion for clock delay and skew minimization, or L_2, as in the design of fixed wireless networks. The MSPT problem is NP-hard even under these restrictions [8]. While for arbitrary metric spaces the $\ln k$-approximation algorithm of Guha and Kuller [4] is best possible unless NP $\subseteq$ TIME$(n^{\log\log n})$

(cf. combined results of [5] and [3]), optimal approximation results are not yet known for the rectilinear and Euclidean planes.

Recently, Lin and Xue [6] considered the following *MST heuristic* for the MSPT problem: Compute an MST on terminals, then subdivide each edge (u, v) of the MST via $\lceil d(u, v)/R \rceil - 1$ equally spaced Steiner points, where $d(u, v)$ stands for the distance between u and v, and $R > 0$ is the prescribed edge-length upper-bound. Lin and Xue proved that the MST heuristic has an approximation factor not worse than 5 in the Euclidean plane, leaving open the problem of finding the exact approximation factor.

We give a tight analysis of the MST heuristic for any L_p metric space, showing that its approximation factor is exactly one less than the *MST number*, defined as the maximum possible degree of a minimum-degree MST spanning points from the space. Since the MST numbers for the rectilinear and Euclidean planes are 4 and 5 [7], our analysis implies that for these two metric spaces the MST heuristic has tight approximation factors of 3 and 4, respectively.

The factor of 4 for the Euclidean plane has been obtained independently by the authors of [2]. The analysis in [2] relies heavily on properties specific to the Euclidean plane and does not seem to extend to other metric spaces. In contrast, our analysis comes closer to the simplicity of the original argument of Lin and Xue [6], using only triangle inequality and the fact that every set of points from the space has an MST with maximum degree no larger than the MST number.

2 Analysis of the MST heuristic

Let (X, d) be a metric space, and let $\tau(P)$ denote the set of all d-weighted MSTs spanning $P \subseteq X$. Following Robins and Salowe [7], the *MST number* of X, $D(X)$, is defined by

$$D(X) = \sup_{P} \min_{T \in \tau(P)} \max_{v \in P} \deg_T(v), \tag{1}$$

where the supremum in (1) is taken over all finite subsets P of X. Note that, if $D(X)$ is finite, then every set of points in X admits an MST with maximum degree at most $D(X)$.

Theorem 1 *The MST heuristic has an approximation factor of $D-1$ in every metric space whose MST number is $D < \infty$.*

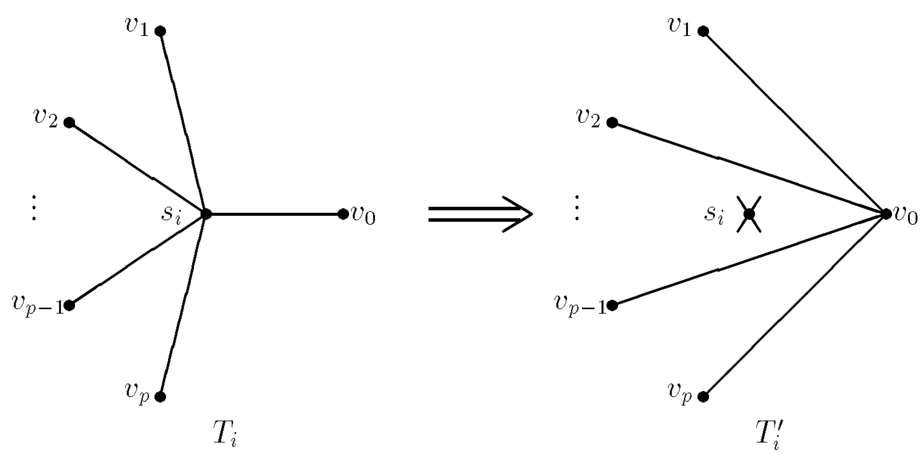
Proof. Let P be a set of terminal points, and let T_{opt} be an MSPT for P. Let $s_1, \ldots, s_k$ be the Steiner points spanned by T_{opt}, numbered in the order in which a breadth-first traversal (started from an arbitrarily terminal $t_0 \in P$) encounters them. Since all edges of T_{opt} have length at most R, it follows that, for every $1 \leq i \leq k$, s_i is within a distance of R of at least one point from $P \cup \{s_1, \ldots, s_{i-1}\}$.

For a tree T, let $beads(T) = \sum_{(u,v) \in E(T)} (\lceil d(u,v)/R \rceil - 1)$ denote the number of subdivision points, or *beads*, that need to be added on T's edges in order to satisfy the edge-length condition. It is easy to see that any MST has minimum number of beads among trees spanning the same set of points; we will use this fact below.

For $1 \leq i \leq k$, let T_i be an MST on $P \cup \{s_1, \ldots, s_i\}$ with maximum degree at most D. We claim that, for every $1 \leq i \leq k$,

$$beads(T_{i-1}) \leq beads(T_i) + (D-1). \tag{2}$$

Let $v_0, v_1, \ldots, v_p$ be the $p + 1 \leq D$ nodes adjacent to s_i in T_i, one of which, say v_0, must be a closest neighbor of s_i in $P \cup \{s_1, \ldots, s_{i-1}\}$. Let T_i' be the tree obtained from T_i by removing s_i and connnecting to v_0 the nodes $v_i, i = 1, \ldots, p$.



Note that $d(s_i, v_0) \leq R$, since the BFS numbering ensures that s_i is within a distance of R of at least one point from $P \cup \{s_1, \ldots, s_{i-1}\}$ and v_0 is the point from this set closest to s_i. By triangle inequality, any edge (v_j, v_0) needs at most one more bead than the edge (v_j, s_i). Hence,

$$beads(T_i') \leq beads(T_i) + p \leq beads(T_i) + (D-1).$$

Inequality (2) follows by noting that $beads(T_{i-1}) \leq beads(T_i')$, since T_{i-1} is an MST spanning the same set of points as T_i'.

Adding inequalities (2) for $0 \leq i \leq k$ and using the fact that $beads(T_k) = 0$ gives $beads(T_0) \leq k \cdot (D-1)$. Thus, the MST on P uses at most $D - 1$ times more Steiner points than T_{opt}. □

Theorem 2 *The approximation guarantee given in Theorem 1 is tight for any fixed-dimensional L_p metric space.*

Proof. Robins and Salowe [7] show that in L_p metric spaces the MST number is finite, being equal to the maximum number of points that can be placed on the surface of a unit ball such that each pair of points is strictly more than one unit apart. When the MST heuristic is run with $R = 1$ on a set of D points realizing the above configuration, the result is a tree with $D - 1$ Steiner points, all of degree 2. On the other hand, the MSPT uses only one Steiner point, of degree D, namely the center of the ball. □

Since the MST number is 4 (resp. 5) for the rectilinear (resp. Euclidean) planes [7], Theorems 1 and 2 give:

Corollary 3 *The MST heuristic has a tight approximation factor of 3 in the rectilinear plane, and of 4 in the Euclidean plane.*

3 Conclusion and open problems

The obvious open problem is to find approximation algorithms that achieve better factors than the MST heuristic in the rectilinear and Euclidean planes. We believe this could be done by an adaptation of the techniques in [9,1], based on restricted Steiner trees. [1]

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